Astrometric Accuracy Limits Due to Tilt Anisoplanatism

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ABSTRACT

The limit on astrometric accuracy due to tilt anisoplanatism is evaluated using a geometric optics analysis. The one-axis, one-sigma error is $1.71/\sqrt{T}$ milli arc seconds for the case of a single reference star separated from the science object by 30 arc seconds, where T is the length of the science exposure in seconds. The error is reduced to $0.27/\sqrt{T}$ milli arc seconds in the case of three reference stars arranged in an equilateral triangle with a common reference-to-target separation of 30 arc seconds for all three stars. These results have been computed for the median $C_n^2(h)$ and wind velocity profiles at Armazones and a turbulence outer scale L_0 of 30 meters.

1. INTRODUCTION

The current SRD requirement for the astrometric accuracy achievable using NFIRAOS is understood to be something of a placeholder, and the TMT SAC and NFIRAOS design team are now working to develop a revision which will be traceable to TMT scientific objectives, the performance demonstrated by existing AO systems, and the limitations likely to be imposed by fundamental and practical error sources. One of these error sources is tilt anisoplanatism, which will cause the apparent separation between the science object and astrometric reference stars to fluctuate with the instantaneous value of the turbulence profile. This error may be reduced by a combination of (i) averaging over time and (ii) averaging over multiple reference stars. Since NFIRAOS nominally uses three natural guidestars to correct both tip/tilt and tilt anisoplanatism, it is hoped that improved astrometric precision may be achieved in relatively short exposure times if these three stars are used as astrometric references.

In this note we evaluate the RMS astrometry errors due to the time-averaged effect of tilt anisoplanatism using a classical geometric optics analysis. Increasing the number of reference stars from 1 to 3 reduces the residual RMS error by about a factor of 7, at least for the case of (i) nominal atmospheric conditions at Armazones, (ii) a 30 m turbulence outer scale, and (iii) a separation of 30 arc seconds between the science field and the astrometric reference star(s). For 3 reference stars the one-axis, one-sigma astrometric error is about $0.27/\sqrt{T}$ milli-arc seconds, where T is the length of the exposure time in seconds. For $T = 100$ seconds, this error is reduced to about 0.4 per cent of the FWHM of a diffraction-limited point source image at a wavelength of 1 micron. Roughly 70 minutes would be required to obtain the same accuracy with only a single reference star.

The fundamental limit on astrometric accuracy imposed by this error source consequently appear to be acceptable when all three of the tip/tilt stars used by NFIRAOS can be used as astrometric references. Much longer integration times would still be required, however, if their coordinates were unknown and had to be determined by NFIRAOS itself. A range of other error sources also remain to be evaluated, including imperfect correct/calibration of atmospheric refraction, data processing errors (particularly in crowded fields), and implementation errors such as uncalibrated optical distortion. The final SRD specification for the astrometric accuracy achievable using NFIRAOS will need to take all of these effects into account.

The remainder of this note is organized as follows: Section 2 contains a summary of the analytical formulas which have been developed, Section 3 summarizes the numerical parameters evaluated, and Section 4 summarizes the results obtained.

2. SUMMARY OF ANALYSIS

The mean-square OPD associated with the time-averaged value of tilt anisoplanatism may be written in the form

$$\sigma^2 = \left\langle \sum_{i=2}^{3} \left\{ \frac{1}{\pi R^2} \int dr\, A(r) Z_i(r/R) \left\{ \frac{1}{T} \int_{-T/2}^{T/2} dt \left[\sum_{j=1}^{m} w_j \mathrm{OPD}_j(r;t) \right] \right\} \right\}^2 \right\rangle . \tag{1}$$

Here R is the telescope aperture diameter, Z_2 and Z_3 are the tip and tilt Zernike modes, and T is the integration time. The functions $\mathrm{OPD}_j(r;t)$ are the turbulence-induced OPD profiles in the direction of the science object ($j = 1$) and the astrometric reference stars ($j = 2, \dots, m$). The weights w_j describe how the tip/tilt measurements in the directions of the $m - 1$ reference stars are combined to estimate the tip/tilt in the direction of the science object. For example, $m = 2$ and $w = (1, -1)$ for the case of a single reference star, and $m = 4$ and $w = (1, -1/3, -1/3, -1/3)$ for the case of three reference stars arranged in an equalateral triangle centered at the science object. The angle brackets $\langle \dots \rangle$ denote ensemble averaging over both atmospheric turbulence statistics and the direction of the wind.

Assuming geometrical optics and and the Taylor (or frozen flow) hypothesis, the optical path difference maps OPD_j are related to the atmospheric refractive index profile $n(r, z)$ by the expression

$$\mathrm{OPD}_j(r;t) = \int_0^\infty dz\, n[r + z\theta_j - tv(z), z]. \tag{2}$$

Here z denotes range along the telescope optical axis, θ_j is the angular offset to source number j, and $v(z)$ is the wind velocity at range z. It is convenient to assume that the windspeed is fixed, but the direction of the wind is uniformly and randomly distributed at each z.

The usual analytical methods are now applied. On account of the Fourier shift theorem, the expression for the mean-square OPD σ^2 may be reduced to the form

$$\sigma^2 = 2\pi \int_0^\infty dz \int_0^\infty \kappa d\kappa\, \Phi(\kappa; z) f_Z(\kappa) f_T(\kappa; z) f_A(\kappa, z) \tag{3}$$

where κ is a one-dimensional spatial frequency variable, Φ is the power spectrum of the refractive index variations n, and the functions f_Z, f_T, and f_A are filter functions associated with the tip/tilt Zernike modes, temporal averaging, and reference star averaging, respectively. For a von Karman spectrum, the refractive index PSD is given by the formula

$$\Phi(\kappa; z) = \frac{0.0097 C_n^2(z)}{(\kappa^2 + 1/L_0^2)^{11/6}}, \tag{4}$$

where L_0 is the outer scale of turbulence and $C_n^2(z)$ is the refractive index structure constant at range z. The tip/tilt mode filter function is given by

$$f_Z(\kappa) = \left[\frac{4 J_2(2\pi R\kappa)}{2\pi R\kappa} \right]^2, \tag{5}$$

and the temporal averaging filter takes the form

$$\begin{aligned} f_T(\kappa; z) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \left\{ \frac{\sin[\pi T\kappa v(z)\cos\theta]}{\pi T\kappa v(z)\cos\theta} \right\}^2 \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n [\pi T\kappa v(z)]^{2n}}{(2n+1)(2n+2)(n!)^2} \\ &\approx \frac{1.014}{\pi T\kappa v(z)} \qquad \text{if } \pi T\kappa v(z) \geq 20. \end{aligned} \tag{6}$$

The remaining filter function, which averages over multiple reference stars, is given by the expression

$$f_A(\kappa; z) = \sum_{j,k} w_j w_k J_0(2\pi\kappa|\theta_j - \theta_k|z). \tag{7}$$

This equation reduces to the simpler form $2 - 2J_0(2\pi\kappa\theta z)$ in the standard case of a single reference star offset by an angle of θ from the science object.

Finally, the mean-square OPD σ^2 may be converted into the corresponding one-axis, one-sigma tip/tilt error using the expression

$$\sigma_\theta = (2/R)\sqrt{\sigma^2/2}. \tag{8}$$

3. SUMMARY OF PARAMETERS

The numerical parameters required for these computations include (i) atmospheric turbulence and wind velocity profiles, (ii) a telescope aperture diameter, (iii) the integration time for the science observation, and (iv) the asterism of reference stars used for the astrometric measurement. The values used for these quantities are summarized in the following paragraphs.

The atmospheric turbulence and wind velocity profiles are summarized in Table 1 below. The $C_n^2(h)$ profile represents the median atmospheric conditions measured at Armazones during the TMT site survey.[2] The wind velocity profile corresponds to median conditions (as measured by daily thermosondes) at the nearby Antofagasta airport.[3] We have evaluate performance for outer scales (L_0) of 30 meters and infinity. These values are thought to represent "typical" and worst case conditions, respectively.

All calculations have assumed a telescope aperture diameter of 30 meters (what else?). Residual atmospheric image distortion has been evaluated for exposure times between 1 and 100 seconds, by which point the $t^{-1/2}$ scaling law is very apparent.

Case and 0, 1, or 3 reference stars have been considered. The separation between the science object and the reference star(s) is always 30 arc seconds, with the stars arranged in an equalateral triangle in the 3-reference-star case.

Table 1. Atmospheric profile summary

Layer number	Altitude, km	Integrated C_n^2	Windspeed, m/s
1	0.0	171.4×10^{-15}	5.47
2	0.5	15.9×10^{-15}	5.92
3	1.0	3.2×10^{-15}	6.21
4	2.0	15.9×10^{-15}	7.63
5	4.0	28.6×10^{-15}	11.52
6	8.0	44.4×10^{-15}	24.99
7	16.0	38.1×10^{-15}	11.01

4. SUMMARY OF RESULTS

Figure 1 plots sample results for the astrometric error due to unaveraged atmospheric image distortion as a function of the science exposure integration time. The $t^{-1/2}$ scaling of the error with integration time is readily apparent, as is the dependence of the astrometric error upon the value of the turbulence outer scale L_0. Increasing the number of reference stars from 1 to 3 consistently reduces the error by factors of 7 to 10, which correspond to factors of 50 to 100 in the required integration time for equivalent accuracy. The astrometric errors with 3 reference stars are consistent with the current specification in the TMT SRD,* while the specification is not met with only a single reference star.

*A residual error of 1 per cent of the diffraction-limited PSF FWHM after an integration time of 100 seconds, or about 70-150 mas depending upon the wavelength.

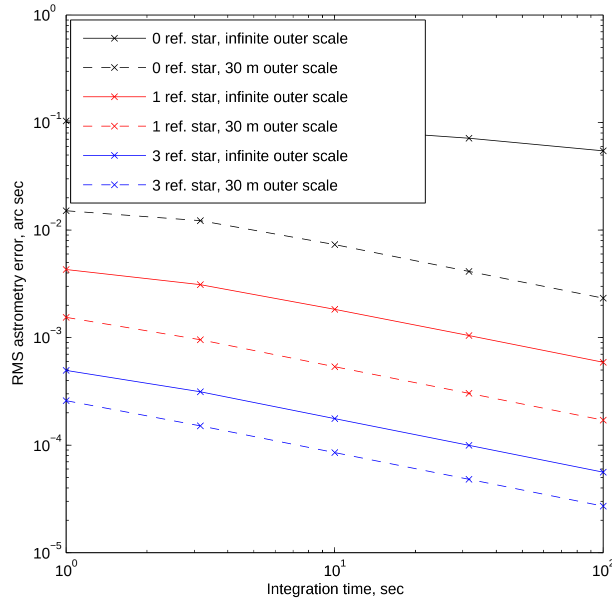

Figure 1. Astrometry errors due to atmospheric image distortion as a function of integration time, number of reference stars, and turbulence outer scale. See the text for further description of the parameters.

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